A Gentle Introduction to Grassmannians

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Abstract

We introduce the Grassmannian $G(k, V)$ as a set, and via the tools of multilinear algebra, the Plücker embedding and the Plücker coordinates on $G(k, V)$. We also prove some basic theorems about the Grassmannians and attempt to familiarize the reader with some of the fundamental ideas and language needed to work with the Grassmannian and its subvarieties.

1 Introduction and Motivation

The Grassmannian is a fundamental object of study across various subdisciplines of modern geometry. They are not only crucial in constructing other interesting varieties but have a rich structure and are of basic interest in their own right. Historically, the space dates back to Plücker, who studied the space of all projective lines in the space $\mathbb{P}^3$, the space $G(2, 4)$. In general the Grassmannian $G(k, V)$ as a set is simply the set of all $k$-dimensional linear subspaces in $V$. Thus, it naturally is a space parametrizing linear subspaces of a vector space and and is useful for enumerative problems. The (real or complex) Grassmannian can also be given the structure of a differentiable manifold and studied from a differential geometry perspective. In fact, there has been work in Gromov-Witten theory (a field of modern mathematical string theory) on the complex Grassmannian. In this paper, we will focus on the algebraic structure of the Grassmannian and in particular, we will realize it as a projective variety.

Recall that in the case of $k = 1$, this is the familiar construction of projectivization of a vector space, $\mathbb{P}V$. Given this set we associate the points in $\mathbb{P}V$ with the solutions of homogenous polynomials, and hence give projective space the structure of a variety. In what follows, we will attempt to do the same with the Grassmannian, and understand it as a subvariety of a projective space of sufficiently large dimension. Again we define the Grassmannian $G(k, V)$ as

$$G(k, V) := \{ W \subset V : W \text{ is a subspace, } \dim W = k \}.$$

To understand this we will introduce briefly the language of the wedge product and multivectors.
2 A Crash Course in Multilinear Algebra

Multilinear algebra is the study of functions that are linear in multiple variables, that is functions from $V_1 \times V_2 \times \cdots \times V_n \to V$, linear in each of the variables. However, in this paper we will focus on defining the crucial properties required to understand the Grassmannian structure. What follows are the definitions of some basic terms and constructions that will be used in the proof the the main results.

**The Wedge Product** The wedge product is an supercommutative product that can be used to generalize certain properties of the familiar cross product in $\mathbb{R}^3$. Given any two vectors $v, w$, we note that $v \wedge w$ is an element of $\bigwedge^2 V$ (to be defined), is a simple 2-multivector, or a blade that satisfies

$$v \wedge w = -w \wedge v.$$

One can easily see now that $v \wedge v = 0$. Similarly, we can define a $k$-multivector, as $v_1 \wedge \cdots \wedge v_k$, noting that interchanging 2 adjacent elements negates the product. \(^1\)

**The Exterior Power of a Vector Space** Given a vector space $V$, the $k$th exterior power of the vector space, $\bigwedge^k V$ is the span of the $k$-blades in $V$, or the simple $k$-multivectors. That is

$$\bigwedge^k V := \text{span}\{v_1 \wedge \cdots \wedge v_k : v_i \in V\}.$$

It is critical for our purposes that the space $\bigwedge^k V$ is a vector space.

**A Note on Total Decomposability and Divisors** A $k$-multivector $\omega \in \bigwedge^k V$ is said to be *totally decomposable* if it can we written as $k$-blade, that is if $\omega = w_1 \wedge \cdots \wedge w_k$. For example, the form multivector $e_1 \wedge e_2 + e_3 \wedge e_4$ is a non-totally decomposable 2-multivector in $\bigwedge^2 V$. (For the interested reader, this is in fact a symplectic 2 form).

We also need to introduce the divisors of a multivector. Given a multivector $\omega \in \bigwedge^k V$, $v \in V$ is said to be a *divisor of* $\omega$ if $\omega$ can be written as

$$\omega = v \wedge \varphi$$

such that $\varphi \in \bigwedge^{k-1} V$.

It is not hard to see that if $\omega$ is a totally decomposable $k$-multivector, its space of divisors is a subspace of dimension $k$. We will use this fact later on in this paper.

\(^1\)For the purpose of transparency, the exterior power of a vector space is an important construction that arises in several algebraic and topological settings including the calculus of differential forms. A reader unfamiliar with this construction is encouraged to approach one of the several classical treatments of the exterior product and the exterior algebra such as [2].

2
3 The Plücker Embedding of $G(k, V)$

Notice that given a point in the Grassmannian, $W \in G(k, V)$, $W$ is of dimension $k$ and is thus generated by $k$ linearly independent vectors in $V$. In fact, give a basis, we can associate to $W$ the multivector $\lambda = v_1 \wedge \cdots \wedge v_k$ where $\{v_i\}$ span the space $W$. We note here that $\lambda$ is determined up to scalars. Choosing a different basis, the corresponding $\lambda$ is scaled by the determinant of the change of basis matrix. Thus, compensating for the scaling invariance, we have a well defined map to the projectivization of $\bigwedge^k V$.

$$\psi : G(k, V) \rightarrow \mathbb{P}\left(\bigwedge^k V\right)$$

$W = \langle v_1, \ldots, v_k \rangle \mapsto [\lambda]$ where $[\lambda]$ is the projectivized coordinate of $\lambda$ defined above i.e. the linear subspace generated by $v_1 \wedge \cdots \wedge v_k$.

This map $\psi$ is known as the Plücker map and is an embedding of $G(k, V)$ into $\mathbb{P}\left(\bigwedge^k V\right)$. To see that $\psi$ is an embedding, we notice that for any $[\omega] = \psi(W)$, $W$ is the space of all vectors $v \in V$ such that $v \wedge \omega = 0 \in \bigwedge^{k+1} V$. It is simple to see here that the image $\psi(G(k, V))$ is the projectivization of the space of all totally decomposable vectors in $\bigwedge^k V$. The coordinates on $\mathbb{P}^N = \mathbb{P}\bigwedge^k V$ are known as the Plücker coordinates on $G(k, V)$. Concretely, choosing an identification $V = \mathbb{K}^n$, we can represent the space $W$ with a $k \times n$ matrix $M_W$ whose rows are formed by the vectors $v_i$; and in this case the Plücker coordinates are just the maximal minors of the matrix.

4 $G(k, V)$ as a Variety

The purpose of this section is to prove, using the Plücker embedding that the Grassmannian $G(k, V)$. We will do this as follows. We will classify all totally decomposable vectors in $\bigwedge^k V$, and these will be precisely the points which are precisely the points in the Grassmannian identified with their injection into $\mathbb{P}^N$. Note that we will henceforth identify the points in the Grassmannian with the Plücker embedding.

Recall as we noted in section 2, that a multivector $\omega$ is totally decomposable if and only if is space of divisors is $k$-dimensional. To each $[\omega]$ in the Grassmannian, associate a map $\varphi_{\omega} : V \rightarrow \bigwedge^{k+1} V$ such that

$$v \mapsto v \wedge \omega.$$  

We can easily see that the divisors are in the kernel of this map and hence, $\omega$ is in the Grassmannian if and only if $\varphi_{\omega}$ has rank at most $n - k$. But observe that once we have made an identification $V$ by choosing a basis, this is simply a determinental condition.
the vanishing of the \((n - k + 1) \times (n - k + 1)\) minors of this matrix of \(\varphi_\omega\). Since the determinental condition is a polynomial condition, homogenizing, we can realize the Grassmannian \(G(k, V)\) as the intersection of (finitely many) projective hypersurfaces, and hence it must be a variety.

References
