The Klein Quartic

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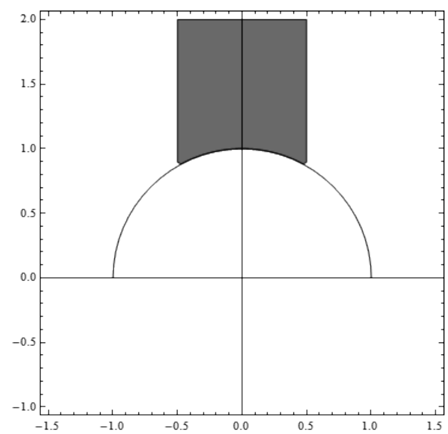
December 14, 2010

1 Introduction

The Klein Quartic curve is famous for its symmetries and many visualizations are created to capture these in lower dimensions. In fact, it has the largest possible automorphism group for its genus, the simple group of order 168. Yet, interest in this algebraic curve started with knowledge of the existence of this large group, and the work that had to be done to express its action on an algebraic curve is often overlooked. Thus, this paper aims to give a historical account of the steps that led to the discovery of the equation of the Klein Quartic surface in projective coordinates.

Figure 1: Fundamental region R of the Modular Group, given by $\{z \in \ H: \ |z| > 1, -\frac{1}{2} \leq Re(z) \leq \frac{1}{2}\}$



2 Background

In response to the work done on elliptic functions by Jacobi and Abel, Dedekind worked on establishing a independent theory of modular functions in 1878 [3]. This led him to study the action of the special linear group $SL(2, \mathbb{Z})$, of 2-dimensional matrices with determinant 1, on H, the upper half-plane of $\mathbb{C}$ defined by

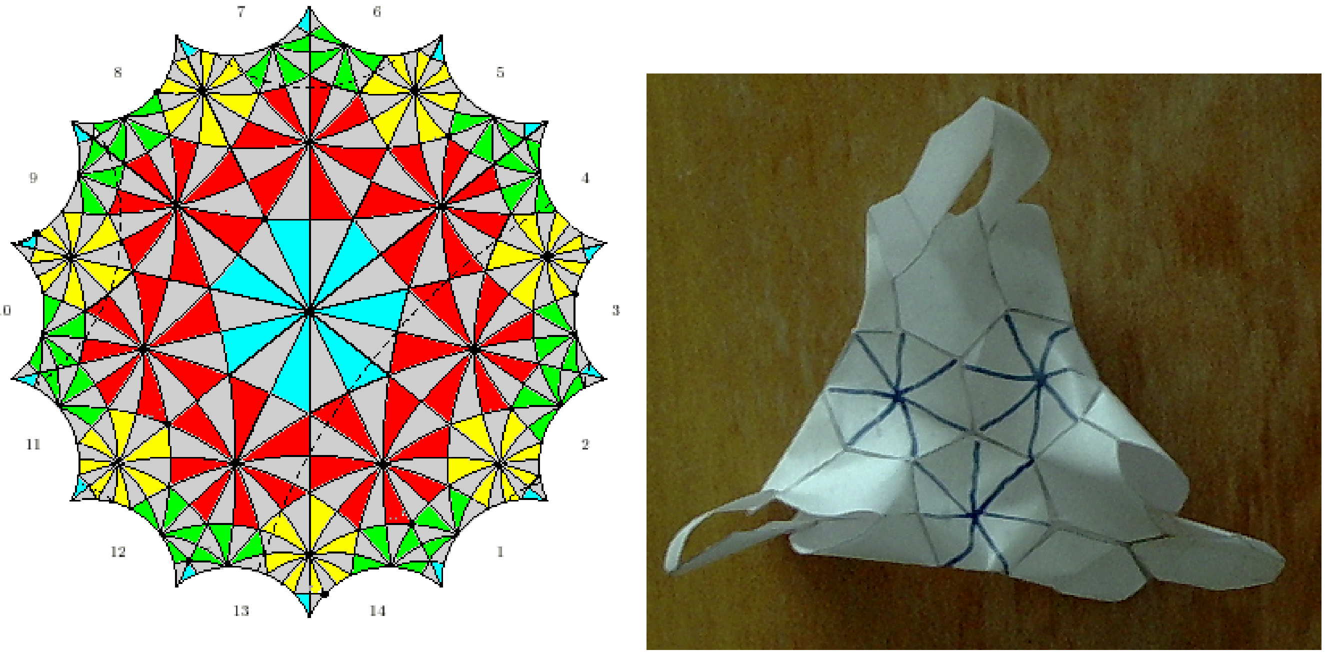
$$z \mapsto \frac{az+b}{cz+d} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \tag{1}$$

He was interested in a fundamental region R where the the orbit of each $z \in H$ meets R either once at the interior or twice at the boundary [3], and he identified R to be the region shown in figure 1. He showed that it was generated by only the two elements $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which inverts the region and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which translates it by 1 to the right [5]. Dedekind concluded that the action of $SL(2, \mathbb{Z})$ tiles H with copies of R.

3 Klein

Upon inheriting this problem, Felix Klein refined its geometric properties by considering its subgroups [3]. One of these was the kernel Γ_7 of the homomorphism

$$SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/7\mathbb{Z}).$$



(a) Klein's surface shown as a tiling of the Riemann sphere.

(b) A paper version of Figure 2a constructed by Mike Stay. `<http://math.ucr.edu/ mike/klein/>`

Figure 2

By Lagrange's theorem, the index of Γ_7 in $SL(2,\mathbb{Z})$ is given by the order $SL(2,\mathbb{Z}/7\mathbb{Z})$. He then considered its action on a set of eight elements and through combinatorial considerations found the order of the group to be 336. Then, since the action of $SL(2,\mathbb{Z})$ given by (1) is the same for matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$, it is useful to quotient out by the center $\{\pm 1\}$ to obtain $PSL(2,\mathbb{Z}/7\mathbb{Z})$ of order 168, which acts faithfully on H. We observe that $\bar{\Gamma}_7$ moves about 168 copies of R through the example in [3]. The property that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \text{shows that} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^7 \in \bar{\Gamma}_7$$

and will therefore move a strip of 7 copies of R to the right. One can picking another element in $\bar{\Gamma}_7$ to obtain more copies of R until a block of 168 is found that is moved around as a whole, and one can choose these R-copies depending on what is needed [3].

We know by the first isomorphism theorem that $\bar{\Gamma}_7$ is normal, so we can consider the quotient $G_{168} = PSL(2,\mathbb{Z})/\bar{\Gamma}_7 \cong PSL(2,\mathbb{Z}/7\mathbb{Z})$. Since its elements are in $SL(2,\mathbb{Z})$, they also move R around, but the action is only defined modulo 7 so G_{168} really maps the block to itself, permuting about copies of R. Thus, we can consider the quotient $H/\bar{\Gamma}_7$ as a curve, while G_{168} gives the automorphism group for this curve [1].

However, in projective coordinates, the region in Figure 1 will have a vertex at ∞. In order to deal with this issue, Klein transformed R to a region where this vertex had angle $2\pi/7$, which led him to construct Figure 2a. This choice makes sense since $\bar{\Gamma}_7$ cycles 7 copies of R about this vertex [3]. Figure 2a then gives the quotient space $H/\bar{\Gamma}_7$, which is consistent with G_{168}, its group of automorphisms. The figure shows a total of 24 heptagons, with 7 red, yellow and green ones, and three blue ones, with one in the center and two broken ones around the edges. Klein then identified edges 1 and 6, 3 and 8, ..., $2n+1$ with $2n+6$ mod 14, connected in a way that the triangles of the same color never touch [1]. An attempt to do this with paper is shown in Figure 2b.

Klein's goal was then to express the figure as an algebraic curve and as a Riemann surface [3], and he was able to compute its genus using Euler's formula

$$V - E + F = \chi$$

where χ is the Euler characteristic. We can see from Figure 1 that the region R is made up of 2 triangles and since $H/\bar{\Gamma}_7$ contains 168 copies of R, so there are triangles 336 total, as shown in Figure 2a, so we can compute

$$F = 336, \quad \text{and} \quad E = \frac{3}{2}(336)$$

since each triangle has 3 edges but each edge is counted twice. Then, if we look at the angles of each triangle in Figure 2a, we see that there are angles that measure $\frac{\pi}{7}$ as in the center, $\frac{\pi}{3}$ where two heptagons of the same color and one of a different color meet and and $\frac{\pi}{2}$ where two heptagons of different colors share an edge. Thus, we obtain

$$V = 366\left(\frac{1}{2\pi}\frac{pi}{7} + \frac{1}{2\pi}\frac{\pi}{3} + \frac{1}{2\pi}\frac{\pi}{2}\right) = 336\left(\frac{1}{14} + \frac{1}{6} + \frac{1}{4}\right) = 164$$

giving $\chi = -4$ which corresponds to a genus of 3. Indeed, if we look at the paper model in Figure 2b, we can sort of see the genus 3 property, as well as the tetrahedral symmetry we would expect from a surface with automorphism group of order 168. Thus result enabled Klein to use Riemann's inequality, a result from 1857 that connected the complex analysis of a connected, compact Riemann to its genus, to conclude that the surface in question can be described by a quartic in $\mathbb{P}^2$ [3]. His remaining work then relied on machinery developed for studying plane curves, leading him to construct an equation for the surface.

4 Plane Curve Machinery

Historically, the systematic study of algebraic curves remained unexplored by mathematicians until the nineteenth century. In contrast to the study detailed analysis of conics that took places in the 1600's, mathematicians found it difficult to generalize the properties of higher-degree curves. Klein's remaining work relied heavily on the results by Julius Plücker, developed in 1835, whose work gave general counting arguments to enumerate the points of inflection, bitangents and cusps of higher-order curves that do not appear on conics [3]. These arguments, along with his group of automorphisms, allowed Klein to develop the equation of his **symmetrical thinger.**

4.1 Inflection Points

One useful result shown by Plücker in 1835 and then proved more simply by Hesse in 1844 showed that a nonsingular curve $C : (F(x, y, z) = 0)$ of degree n has $3n(n-2)$ inflection points [3]. This makes sense, since the determinant of the Hessian matrix of second partials vanishes at an inflection point. Such a matrix would have dimension 3×3 in $\mathbb{P}^2$ and for an algebraic curve of degree n, the second partials would have degree $n-2$. This condition gives a curve of degree at most $3(n-2)$, which we must intersect with $F = 0$ to find the inflection points of C. Then, by Bezout's Theorem, we know that these curves will intersect at $3n(n-2)$ points and these intersections give the number of inflection points.

4.2 Bitangents

Another useful result by Plücker relates the number of bitangents of an algebraic curve to its degree [3]. This result comes from constructing the dual of a curve, which is formed by the set of lines tangent to the original curve. If we consider a curve $C : (F(x, y, z) = 0)$, then the equation of the line tangent to C at a point p is given by

$$x\frac{\partial F}{\partial x}(p) + y\frac{\partial F}{\partial y}(p) + z\frac{\partial F}{\partial z}(p) = 0.$$

We can also consider

$$\left(\frac{\partial F}{\partial x}(p), \frac{\partial F}{\partial y}(p), \frac{\partial F}{\partial z}(p)\right)$$

to define the coordinates of the tangent. However, we can also think of this triple as a point in the dual space, the space of all linear functions. The set of all such points defines a new plane curve called the dual curve of C.

The polar of a curve C at a point $p = (p_x, p_y, p_z)$ is given by

$$\Delta_p = p_x \frac{\partial F}{\partial x}(x,y,z) + p_y \frac{\partial F}{\partial y}(x,y,z) + p_z \frac{\partial F}{\partial z}(x,y,z) = 0$$

and $C*$. We can consider the dual $C*$ as a sort of bucket which contains, for every point in C, a point corresponding to the line tangent to C at that point. The dual of C can be computed by taking the envelope of its polars [3]. Therefore, we expect that the dual of C will be of the same degree as its polar; that is, of degree $n-1$. Again, we apply Bezout's Theorem in order to obtain the number of tangent lines to C, which gives $n(n-1)$.

It turns out that taking the dual of the dual of a curve gives back the original curve. Yet, the above formula suggests that the degree of this curve should be given by

$$n(n-1)(n(n-1)-1),$$

which is not true for $n > 2$. The resolution to this paradox is given by Plücker, who realizes that that correspondence between points and tangent lines is not one-to-one. If we consider a bitangent, a line tangent to two distinct points on C, then such a line will dualize to two points on $C*$, according to the bucket analogy. These points will have two different tangents in the dual space, giving two extra points when $(C*)*$ is computed. A similar situation arises from the existence inflection points on C. An inflection point of C will dualize to a cusp of $C*$, which will dualize again to multiple points. Therefore, if a curve C has α double points and β inflection points, then we expect that

$$2\alpha + 3\beta = n(n-1)(n(n-1)-1) - n = n^3(n-2).$$

If we take our previous result that $\beta = 3n(n-2)$, then we obtain

$$\alpha = \frac{1}{2}n(n-2)(n^2-9).$$

Thus, that we now know to that Klein would expect to find 24 inflection points and 28 bitangents on a non-singular curve of degree 4 in $\mathbb{P}^2$. Indeed, many other mathematicians were interested in quartics as well; for example, Plücker was able to find a non-singular quartic whose bitangents were real. Jordan identified the automorphism group of these bitangents in 1870, and showed that the subgroup that of the group of symmetries that fixes a bitangent is isomorphic to the group of symmetries of the 27 lines on a cubic surface [3].

5 Equation

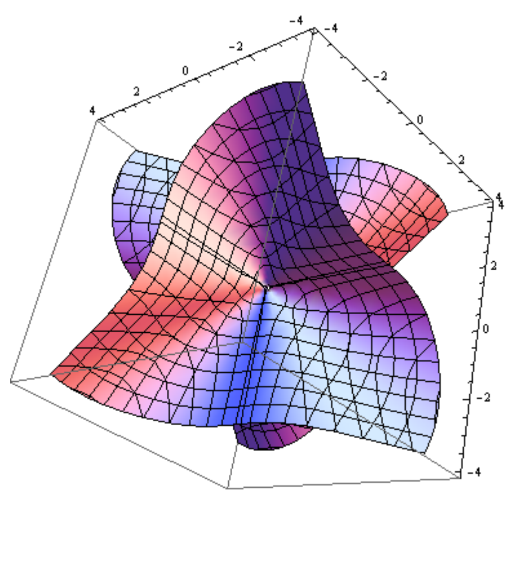
Most descriptions of the Klein quartic give the equation and proceed to discuss its symmetries; however, it turns out that this is precisely backwards. In fact, Klein began with a group-theoretic description of the quartic, and had been aware of its symmetries. Meanwhile, a lot of geometric scaffolding had to be put into place for Klein to utilize it when searching for his equation.

These results allowed Klein to show that if he could locate three suitable inflection tangents, to create a "triangle of reference" about the curve then the curve is given by the equation

$$x^3y + y^3z + z^3x = 0. \tag{2}$$

While most points have orbits of 168 points under the action of G_{168}, Klein studied the points with orbits consisting of 24, 56

Figure 3: 3D plot of the real locus of equation (2) oriented to emphasize a triangle of reference that could be drawn around its folds.



and 84 points. He claimed that these must correspond to the 24 inflection points, 56 bitangent points (since 28 bitangents are tangent at two points each) and 84 sextatic points, where a conic has contacts the curve in 6 points [3].

In order to show this, Klein undertook a thorough examination of the subgroups of G_{168}, finding, among others, 8 conjugate subgroups of order 21 and two families of 7 conjugate subgroups of order 24. He argued that the bitangent points were fixed by a rotation of order 3, which corresponds to the 28 subgroups of order 3 in G_{168}. Similarly, he claimed that the sextatic points are fixed by rotations of order 2, of which there are 21, so each rotation fixes 4 of them [3]. This analysis allowed Klein to provide visual descriptions of the surface.

6 Additional Comments

Therefore, the history of the Klein curve often does not correspond to the order in which it is commonly presented. A lot of serious geometric considerations went into its expression as an algebraic curve, combining many results from analysis, algebraic geometry and group theory. And even though the equation is known, the Klein quartic continues to appear in number theory, revealing many interesting properties.

Another thing to point out is that Klein's study of the group $SL(2, \mathbb{Z}/7\mathbb{Z})$ was not accidental, since he had previously undertaken a similar analysis of the symmetries of an icosahedron, which is acted upon by a simple group isomorphic to $SL(2, \mathbb{Z}/5\mathbb{Z})$ of order 60. The group of order 168 is the second smallest non-abelian simple group, while the icosahedron group is the smallest.

Finally, it is interesting to note that the Klein quartic has the maximal symmetry possible for its genus. Further work on such surfaces undertaken by Hurwitz revealed the bound on the order of the automorphism group for algebraic curves with genus g to be given by the equation $84(g-1)$ [4].

References

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