Kummer Surfaces

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A Kummer surface is a special type of quartic surface. As a projective variety, a Kummer surface may be described as the vanishing set of an ideal of polynomials. In fact, it is the vanishing set of a single polynomial (a principally generated ideal). However, these surfaces may also be viewed more abstractly, in terms of Jacobian varieties. In Section 1 we will state the defining equation and state some interesting properties of Kummer surfaces. In Section 3, we will give an overview of the more abstract approach. In order to do this, we will need a basic discussion of divisors on curves, which is given in Section 2

1 Kummer surfaces

We begin by defining a Kummer surface in terms of a homogeneous quartic polynomial. This equation is the one used in [2] and [5]. Other authors, however, prefer different (equivalent) formulas. One such alternative is used in [1].

Definition 1. *The* Kummer surface *with parameter* $\mu \in \mathbb{R}$ *is the projective variety given by*

$$K_\mu \colon \Big((x^+ y^2 + z^2 - \mu^2 w^2) - \lambda pqrs = 0 \Big) \subset \mathbb{P}^3,$$

where $\mu^2 \neq \frac{1}{3}, 1, 3,$

$$\lambda = \frac{3\mu^2 - 1}{3 - \mu^2}$$

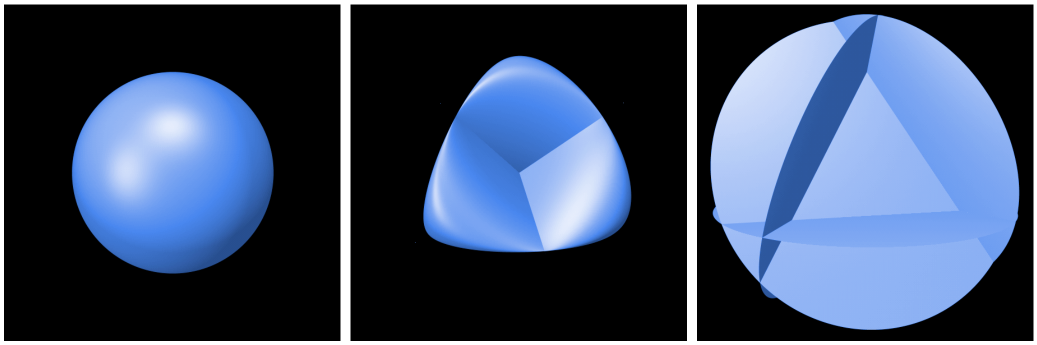

Figure 1: Real plots of the three exceptional cases. From left to right, $\mu^2 = \frac{1}{3}$ (double sphere), $\mu^2 = 1$ (Roman surface), $\mu^2 = 3$ (4 planes). Pictures taken from [2].

and p, q, r, s are the "tetrahedral coordinates," given by

$$\begin{aligned} p &= w - z - \sqrt{2}x, \\ q &= w - z + \sqrt{2}x, \\ r &= w + z + \sqrt{2}y, \ \textit{and} \\ s &= w + z - \sqrt{2}y. \end{aligned}$$

We exclude the values $\mu^2 = \frac{1}{3}, 1, 3$ because these are exceptional cases, for which most of the statements we will make about Kummer surfaces K_μ will not hold. These three cases correspond, repectively, to the double sphere, the Roman surface, and 4 planes, and are shown in Figure 1. As a side note, recall the Veronese surface, which is given by the embedding $\mathbb{P}^2 \to \mathbb{P}^5$ by

$$(x : y : z) \mapsto (x^2 : y^2 : z^2 : xy : xz : yz).$$

The Roman surface referred to above is the projection of this surface into $\mathbb{P}^3$.

We have the following facts about the Kummer surface K_μ.

- K_μ is irreducible.
- As suggested by the appearance of the tetrahedral coordinates in the defining equation, K_μ has tetrahedral symmetry [2].
- K_μ has 16 singularities, each of which is an *ordinary double point*, a three-dimensional analoge of a node singularity. It is interesting to note

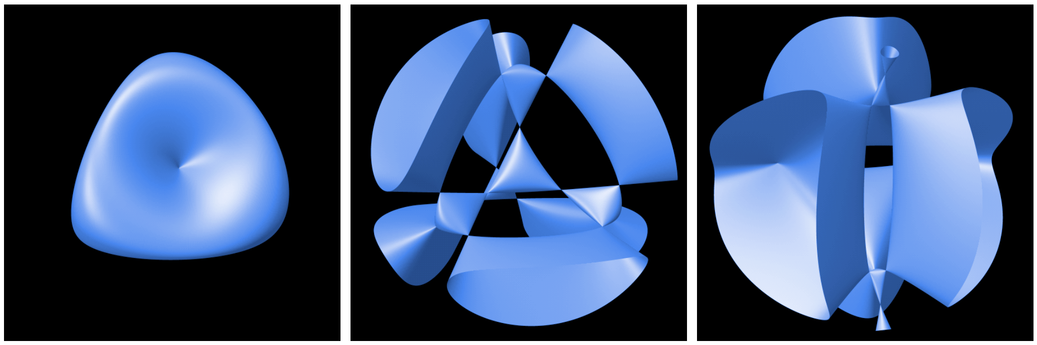

Figure 2: Some Kummer Surfaces. From left to right, the values of the parameter are $\frac{1}{3} < \mu^2 < 1$, $1 < \mu^2 < 3$, and $\mu^3 > 3$. Note that in the surface on the left, not all of the 16 singularities are real. Pictures taken from [2]

that a complex surface may have at most finitely many ordinary double points. For quartic surfaces, the maximum number of ordinary double points is 16, which is achieved by K_μ [2].

- Resolving the 16 singularities of K_μ, we obtain a K3 surface. This K3 surface, which is sometimes given as the definition of a Kummer surface, contains 16 disjoint rational curves [2].

Some Kummer surfaces are shown in Figure 2. Note that since these plots are restricted to the real numbers, some of the ordinary double points may not be visible. In fact, in the case $0 \leq \mu^2 \leq \frac{1}{3}$ (not plotted), K_μ only contains 4 real points (each of which turns out to be an ordinary double point) [2].

2 Divisors

We now give a brief introduction to divisors on curves. Throughout this section, let C be a nonsingular curve.

Definition 2. *A* divisor *D on C is a finite formal sum of the form*

$$D = \sum_{P \in C} \lambda_P P,$$

where each $\lambda_P \in \mathbb{Z}$, with only finitely many nonzero. In other words, D is an element of the free abelian group over the points of C. We say that a divisor D is effective *if $\lambda_P \geq 0$ for all $P \in C$.*

We define the degree *of the divisor D to be the sum*

$$\deg D = \sum_{P \in C} \lambda_P.$$

The set of divisors on C, denoted $\mathrm{Div}(C)$, forms an abelian group. The set of all divisors of degree zero, $\mathrm{Div}^0(C)$, is a subgroup.

Let f be a rational function defined on the points of C. For any point $P \in C$ at which f has a zero or a pole, define the order of f at C, denoted $\mathrm{ord}_P(f)$, to be the multiplicity of the zero or pole, counted positively for a zero and negatively for a pole (so the order at a zero of multiplicity 3 is 3, while the order at a simple pole is -1). At all other points, set $\mathrm{ord}_P(f) = 0$. The function f defines a divisor,

$$(f) = \sum_{P \in C} \mathrm{ord}_P(f) P. \tag{1}$$

Any divisor of the form (1) is called a *principal* divisor. The set of principal divisors on C, which we will denote as $\mathrm{Prin}(C)$, forms a subgroup of $\mathrm{Div}^0(C)$. We now define the *Picard group*

$$\mathrm{Pic}(C) = \mathrm{Div}(C) / \mathrm{Prin}(C),$$

which has the subgroup

$$\mathrm{Pic}^0(C) = \mathrm{Div}^0(C) / \mathrm{Prin}(C).$$

The purpose of this section was to arrive at the definition of $\mathrm{Pic}^0(C)$, which will make an interesting appearance in Section 3. However, divisors on curves are versatile tools in algebraic geometry. Moreover, they may be defined in higher dimensions, and there are analogous ways to define principal divisors and the Picard group.

3 Another perspective

Finally, we give an alternate definition of Kummer surfaces.

Definition 3. *A* Kummer surface *is the Kummer variety of the Jacobian variety of a smooth hyperelliptic curve of genus* 2.

We now attempt to understand this definition. A *hyperelliptic curve* is simply a curve C given by

$$C\colon \big(y^2 = f(x)\big),$$

where f is a polynomial of degree $d > 4$. The genus of C is determined by the degree of f: a hyperelliptic curve of degree $d = 2g+1$ or $d = 2g+2$ has genus g [3]. Definition 3 involves curves of genus 2; hence, we will be concerned with hyperelliptic curves for which the degree of f is 5 or 6.

The next step in deciphering Definition 3 is understanding Jacobian varieties. The mathematics involved with Jacobian varieties is beyond the scope of this paper, but we aim to describe the general concepts. To any curve C of genus g, there is a way to associate with C a variety $\mathrm{Jac}(C)$, called the *Jacobian*, that has the following properties.

- The Jacobian variety $\mathrm{Jac}(C)$ is an abelian variety. That is, the points on $\mathrm{Jac}(C)$ form an abelian group under some group law.
- The dimension of $\mathrm{Jac}(C)$ is g. In our case, we are dealing with curves of genus 2, and so the Jacobian will be a surface.
- The abelian group structure of $\mathrm{Jac}(C)$ is isomorphic to $\mathrm{Pic}^0(C)$. This remarkable property (which is independent of the definition of the Jacobian, but is used as motivation for our sketch of the construction below), is called the Abel-Jacobi theorem.

We give a very brief sketch of the construction of the Jacobian variety of a curve C. For a more detailed treatment, see [4]. Define

$$C^{(r)} = C^r/S_r,$$

where $C^r = C \times \cdots \times C$ (r times) and S_r is the symmetric group on r letters. Observe that $C^{(r)}$ is the set of unordered r-tuples of points on C, repetition allowed. We can naturally view any element of $C^{(r)}$, then, as a divisor of degree r. Conversely, any divisor of degree r can be written as a point in $C^(r)$. The motivation for this is that there is a natural bijection between $\mathrm{Pic}^0(C)$ and $\mathrm{Pic}^r(C)$ (equivalence classes of divisors of degree r). Essentially, we can now try to find a variety that "looks" like $\mathrm{Pic}^r(C)$ instead of $\mathrm{Pic}^0(C)$. Doing so, we construct $\mathrm{Jac}(C)$ as a union of subvarieties of $C^{(r)}$, but the details involve a mess of category theory.

There is one more remarkable fact about Jacobians: *every* abelian variety of dimension 2 is the Jacobian variety of a genus 2 curve. Hence, we may change the language in Definition 3 to say that a Kummer surface is simply the Kummer variety of any abelian surface.

Finally, the *Kummer variety* of an abelian variety is obtained by taking the quotient of a variety by the involution $a \mapsto -a$, that is, by identifying each point with its inverse under the group law [1].

It turns out that the set of order-n points (under the group law) on an abelian surface of dimension g is isomorphic to $\left(\mathbb{Z}/nZ\right)^{2g}$ [4]. Hence, in our case, there are $2^{2\cdot 2} = 16$ order-2 points on the Jacobian $\mathrm{Jac}(C)$. These, it turns out, are exactly the 16 singular points on the Kummer surface.

References

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